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Generalizations of the Cauchy and Schur Identities

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One of the generalizations proved is that

$$\sum \frac{p^{k(\pi)}}{z(\pi)} = \binom{p+n-1}{p-1} - \binom{n-1}{p-1},$$

where the summation is over all partitions $\pi = (1^{k_1},..., n^{k_n})$ of n with parts not divisible by p, $k(\pi) = k_1 + \cdots + k_n$ and $z(\pi) = 1^{k_1}k_1! \cdots n^{k_n}k_n!$. $p = 1$ gives the Cauchy identity and $p = 2$ the Schur identity. This identity is itself obtained as a particular case of a more general identity and the proof involves a generalization of certain symmetric functions, called Hall functions, which have played a major role in recent enumeration problems.

Let $\pi = \pi(n)$ denote a partition of n, usually denoted by $(1^{k_1},..., n^{k_n})$, with $k_1 + 2k_2 + \cdots + nk_n = n$, $k(\pi) = k_1 + \cdots + k_n$ and

$$z(\pi) = 1^{k_1}k_1! \cdots n^{k_n}k_n!,$$

then two well-known identities are

$$\sum_{\pi} \frac{1}{z(\pi)} = 1, \tag{1}$$

due to Cauchy (see, e.g., [4, p. 70]) and

$$\sum \frac{2^{k(\pi)}}{z(\pi)} = 2, \tag{2}$$

where the summation is over partitions of n into odd parts only, due to Schur [8]. The work in this note arose from a desire to prove the following generalization of (1) and (2):

$$\sum \frac{p^{k(\pi)}}{z(\pi)} = \binom{p+n-1}{p-1} - \binom{n-1}{p-1}, \tag{3}$$

where the summation is over all partitions of n with parts not divisible by p. In fact, further generalizations will be proved and (3) will result as a corollary to a more general theorem. The method of proof will depend on a new class of symmetric functions introduced in this paper which are of independent interest.

Let $\alpha_1, ..., \alpha_l$ be a set of l variables and $t_1, ..., t_m$ another set of m variables. If ρ is an arbitrary real number, define

$$Q(x; \rho; t_1, ..., t_m) = \prod_{i=1}^{m} \prod_{j=1}^{l} \left(\frac{1 - t_i\alpha_j x}{1 - \alpha_j x}\right)^{\rho}$$

$$= \sum_{r=0}^{\infty} q_r(\rho; t_1, ..., t_m)\, x^r, \tag{4}$$

then the $q_r(\rho; t_1, ..., t_m) = q_r(\rho, \mathbf{t})(r = 1, 2, ...)$ are clearly symmetric functions of degree r in the variables $\alpha_1, ..., \alpha_l$. If $m = 1$ and $\rho = 1$, then these are the symmetric functions recently defined by Littlewood [3] which themselves when $t = 0$ give the complete symmetric functions $h_r(\boldsymbol{\alpha})(r = 1, 2, ...)$ and when $t = -1$ the symmetric functions $q_r(-1)$ $(r = 1, 2, ...)$ defined by Schur [8] to calculate the projective characters of the symmetric group [5, 6]. The $q_r(1, \mathbf{t})(r = 1, 2, ...)$ lead to Hall functions [3], which are used in the calculation of the characters of $GL(n, q)$ [1, 7] and to the enumeration of subgroups of the Abelian group. In fact, the $h_r(\boldsymbol{\alpha})(r = 1, 2, ...)$ can be used to prove (1) and the $q_r(-1)(r = 1, 2, ...)$ were used by Schur to prove (2).

If $s_r = \sum_{i=1}^{\infty} \alpha_i^r (r = 1, 2, ...)$ (power sums), then it is well known that

$$S(x) = \log \prod_{i=1}^{l} (1 - \alpha_i x)^{-1} = \sum_{r=1}^{\infty} \frac{s_r x^r}{r} \tag{5}$$

is a generating function for the s_r. Thus, it follows that

$$Q(x, \rho, \mathbf{t}) = \frac{(\exp \rho S(x))^m}{\exp(\rho S(xt_1)) \cdots \exp(\rho S(xt_m))}$$

$$= \exp\left(\rho \sum_{r=1}^{\infty} \frac{\{m - (t_1^r + \cdots + t_m^r)\}\, s_r x^r}{r}\right)$$

$$= \prod_{r=1}^{\infty} \sum_{k_r=0}^{\infty} \frac{1}{\alpha_r!} \{m - (t_1^r + \cdots + t_m^r)\}^{k_r} \left(\frac{\rho s_r}{r}\right)^{k_r} x^{rk_r}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{\pi(x)} \frac{\rho^{k(\pi)}}{z(\pi)}\, e(\pi, \mathbf{t})\, s(\pi)\right),$$

where

$$\pi(n) = (1^{k_1},..., n^{k_n}), \qquad k(\pi) = k_1 + \cdots + k_n ,$$

$$z(\pi) = 1^{k_1}k_1! \cdots n^{k_n}k_n!, \qquad s(\pi) = s_1^{k_1} \cdots s_n^{k_n},$$

and

$$e(\pi, \mathbf{t}) = \{m - (t_1 + \cdots + t_m)\}^{k_1} \cdots \{m - (t_1{}^n + \cdots + t_m{}^n)\}^{k_n} .$$

Comparing this with (4) thus gives

$$q_n(\rho, \mathbf{t}) = \sum_{\pi(n)} \frac{\rho^{k(\pi)} e(\pi, \mathbf{t})\, s(\pi)}{z(\pi)} . \tag{6}$$

Our main result is

THEOREM. *With the same notation as above*

$$\sum_{\pi(n)} \frac{\rho^{k(\pi)} e(\pi, \mathbf{t})\, s(\pi)}{z(\pi)}$$

is the coefficient of x^n *in*

$$\prod_{i=1}^{m} \prod_{j=1}^{l} \left(\frac{1 - t_i \alpha_j x}{1 - \alpha_j x}\right)^{\rho} .$$

The results we want to prove will now be obtained as corollaries to this theorem by giving particular values to the $t_i (i = 1,..., m)$ and $\alpha_j (j = 1,..., l)$. Before we state these corollaries, further notation is introduced.

Let

$$\prod_{i=1}^{l} (1 + \alpha_i x)^{\rho} = \sum_{r=0}^{\infty} T_r(\boldsymbol{\alpha}, \rho)\, x^r,$$

then the $T_r(\boldsymbol{\alpha}; \rho) = T_r(\alpha_1 ,..., \alpha_l ; \rho) (r = 1, 2,...)$ are symmetric functions of degree r in $\alpha_1 ,..., \alpha_l$ [9]. In fact [9],

$$T_r(\boldsymbol{\alpha}, \rho) = \sum_{i_1 + \cdots + i_m = r} \binom{\rho}{i_1} \cdots \binom{\rho}{i_m} \alpha_1^{i_1} \cdots \alpha_m^{i_m}$$

and so $T_r(\boldsymbol{\alpha}, 1) = a_r(\boldsymbol{\alpha})$ (elementary symmetric functions) and

$T_r(\boldsymbol{\alpha}, -1) = (-1)^r h_r(\boldsymbol{\alpha})$. Further, it is easily proved by the usual methods (see proof of (6) above) that

$$T_r(\boldsymbol{\alpha}, \rho) = \sum_{\pi(r)} \frac{\rho^{k(\pi)} s(\pi)}{z(\pi)} \tag{7}$$

$$= \frac{1}{r!} \det \begin{bmatrix} \rho s_1 & 1 & & & \\ \rho s_2 & \rho s_1 & 2 & \bigcirc & \\ \vdots & \vdots & & \ddots & \\ & & & & r-1 \\ \rho s_r & \cdot & \cdot & \cdot & \rho s_1 \end{bmatrix}. \tag{8}$$

We can now state

COROLLARY 1.

$$\sum_{\pi(n)} \frac{\rho^{k(\pi)} e(\pi, \mathbf{t})}{z(\pi)} = \sum_{i=1}^{n} \binom{n-1}{n-i} T_i(\mathbf{1} - \mathbf{t}; \rho) \tag{9}$$

$$= \sum_{i=0}^{n} (-1)^i \binom{m\rho + n - 1 - i}{n-i} T_i(\mathbf{t}; \rho), \tag{10}$$

where

$$T_i(\mathbf{1} - \mathbf{t}; \rho) = T_i(1 - t_1, \ldots, 1 - t_m; \rho)$$

and

$$T_i(\mathbf{t}; \rho) = T_i(t_1, \ldots, t_m; \rho).$$

Proof. Let $\alpha_1 = 1$, $\alpha_2 = \cdots = \alpha_l = 0$ in the theorem, then $s(\pi) = 1$ for all $\pi(n)$ and so the left-hand side equals the coefficient of x^r in

$$\prod_{i=1}^{m} \left(\frac{1 - t_i x}{1 - x}\right)^\rho.$$

This is now evaluated in two ways. First,

$$\prod_{i=1}^{m} \left(\frac{1 - t_i x}{1 - x}\right)^\rho = \prod_{i=1}^{m} \left(1 + (1 - t_i) \frac{x}{1 - x}\right)^\rho$$

$$= \sum_{s=0}^{\infty} T_s(1 - t_1, \ldots, 1 - t_m; \rho)\left(\frac{x}{1-x}\right)$$

$$= \sum_{s=0}^{\infty} T_s(\mathbf{1} - \mathbf{t}; \rho)\left(x^s \sum_{j=0}^{\infty} \binom{s+j-1}{j} x^j\right),$$

and the coefficient of x^n in this gives (9). Alternatively

$$\prod_{i=1}^{m} \left(\frac{1 - t_i x}{1 - x}\right)^{\rho} = (1 - x)^{-m\rho} \prod_{i=1}^{m} (1 - t_i x)^{\rho}$$

$$= \left\{\sum_{k=0}^{\infty} \binom{m\rho + k - 1}{k} x^k\right\}\left\{\sum_{j=0}^{\infty} (-1)^j T_j(t_1, \dots, t_m ; \rho)\, x^j\right\}$$

and the coefficient of x^n in this gives (10).

Some special cases of this corollary are worth stating separately:

COROLLARY 2.

$$\sum_{\pi(n)} \frac{(1 - t)^{k_1} \cdots (1 - t^n)^{k_n}}{z(\pi)} = 1 - t, \tag{11}$$

$$\sum \frac{(\rho p)^{k(\pi)}}{z(\pi)} = \sum_{i=0}^{\infty} (-1)^i \binom{\rho}{i}\binom{\rho p + n - pi - 1}{n - pi}, \tag{12}$$

$$\sum \frac{p^{k(\pi)}}{z(\pi)} = \binom{p + n - 1}{p - 1} - \binom{n - 1}{p - 1}, \tag{13}$$

where the summation in (12) *and* (13) *is over all partitions of n into parts not divisible by p.*

Proof. (11) results from (9) of Corollary 1 by putting $\rho = 1$, in which case

$$\sum_{\pi(n)} \frac{e(\pi, \mathbf{t})}{z(\pi)} = \sum_{i=1}^{n} \binom{n - 1}{n - i} a_i(\mathbf{1 - t})$$

and if $m = 1$, $a_1(\mathbf{1 - t}) = 1 - t$ and $a_i(\mathbf{1 - t}) = 0$ for $i > 1$. Thus

$$\sum_{\pi(n)} \frac{(1 - t)^{k_1} \cdots (1 - t^n)^{k_n}}{z(\pi)} = 1 - t.$$

(12) is obtained from (10) of Corollary 1 by letting $m = p$ and $t_1 = 1$, $t_2 = \zeta, \dots, t_p = \zeta^{p-1}$, where ζ is a primitive p-th root of unity, in which case the left-hand side in (10) of Corollary 1 reduces to

$$\sum \frac{(\rho p)^{k(\pi)}}{z(\pi)},$$

where the summation is over all partitions of n not divisible by p. By means of (8), we note that, since $s_i = 0$ if i is not a multiple of p and $s_i = p$ if i is a multiple of p, then

$$\begin{aligned} T_j(1, \zeta, ..., \zeta^{p-1}; \rho) &= 0 \qquad \text{if} \quad j \neq ip \\ &= (-1)^{i(p-1)} \qquad \text{if} \quad j = ip. \end{aligned}$$

Thus, it follows from (10) that

$$\sum \frac{(\rho p)^{k(\pi)}}{z(\pi)} = \sum_{i=0}^{n} (-1)^i \binom{p\rho + n - 1 - ip}{n - ip} \binom{\rho}{i},$$

where the summation on the left is over all partitions of n not divisible by p.

(An alternative direct proof for (12) is obtained if it is noted that the left-hand side equals the coefficient of x^n in

$$\prod_{i=1}^{p-1} \left(\frac{1 - \zeta^i x}{1 - x} \right)^\rho = \frac{(1 - x^p)^\rho}{(1 - x)^{p\rho}} \Big).$$

(13) now follows directly for, if $\rho = 1$, then $\binom{\rho}{i} \neq 0$ only if $i = 0$ or 1.

REMARK. (10) may be regarded as a generalization of Sylvester's theorem [4] that

$$\sum_{\pi(n)} \frac{\rho^{k(\pi)}}{z(\pi)} = \binom{\rho + n - 1}{n}, \tag{14}$$

which itself is a generalization of (1). If we put $m = 1$ and $t = 0$ in (10), then $T_i(0; \rho) = 0$ for $i > 0$ and (14) results. It may be further noted that the identity obtained by Harrison and High [2] is simply (14) with $\rho = m$ and $n = m^n$.

REFERENCES

1. J. A. GREEN, The characters of the finite general linear groups, *Trans. Amer. Math. Soc.* **80** (1955), 402–447.
2. M. A. HARRISON AND R. G. HIGH, On the cycle index of a product of permutation groups, *J. Combinatorial Theory* **4** (1968), 277–299.
3. D. E. LITTLEWOOD, On certain symmetric functions, *Proc. London Math. Soc.* (3) **11** (1961), 485–498.
4. P. A. MACMAHON, "Combinatory Analysis," 2 vol., Cambridge University Press, 1915–16; reprinted by Chelsea, New York, 1960.

5. A. O. MORRIS, Spin representation of the symmetric group, *Proc. London Math. Soc.* (3) **12** (1962), 55–76.
6. A. O. MORRIS, On Q-functions, *J. London Math. Soc.* **37** (1962), 445–455.
7. A. O. MORRIS, The characters of the group $GL(n, q)$, *Math. Z.* **81** (1962), 112–123.
8. I. SCHUR, Über die Darstellung der symmetrischen und den alternierenden Gruppe durch gebrochene lineare Substitutionen, *J. reine angew. Math.* **139** (1911), 155–250.
9. J. N. WHITELEY, Some inequalities concerning symmetric forms, *Mathematika* **5** (1958), 49–57.